Lecture 18: More on column space algorithm, null space, extending LI sets, finding and extending bases

November 19, 2016 12:17 AM

16.4 Why does the column space algorithm work

$$\mathrm{Col}\underbrace{\begin{bmatrix}1 & 2 & -1 & 4\\5 & 10 & -4 & 19\\2 & 4 & -2 & 8\end{bmatrix}}_{A} = \mathrm{span}\left\{\begin{pmatrix}1\\5\\2\end{pmatrix},\begin{pmatrix}2\\10\\4\end{pmatrix},\begin{pmatrix}-1\\-4\\-2\end{pmatrix},\begin{pmatrix}4\\19\\8\end{pmatrix}\right\}$$

We'd like to find a basis:

$$\begin{bmatrix}1 & 2 & -1 & 4\\5 & 10 & -4 & 19\\2 & 4 & -2 & 8\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -1 & 4\\0 & 0 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}\sim\underbrace{\begin{bmatrix}1 & 2 & 0 & 3\\0 & 0 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}}_{\text{RREF}}$$

RREF

<u>Basis for the column space</u>: $\left\{\begin{pmatrix}1\\5\\2\end{pmatrix},\begin{pmatrix}-1\\-4\\-2\end{pmatrix}\right\}$

When we want to solve the following problem:

$$x_1\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}+x_2\cdot\begin{pmatrix}2\\10\\4\end{pmatrix}+x_3\cdot\begin{pmatrix}-1\\-4\\-2\end{pmatrix}+x_4\cdot\begin{pmatrix}4\\19\\8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

We obtain the following solution:

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 4 & 0\\5 & 10 & -4 & 19 & 0\\2 & 4 & -2 & 8 & 0\end{array}\right]\sim\ldots\sim\left[\begin{array}{cccc|c}1 & 2 & 0 & 3 & 0\\0 & 0 & 1 & -1 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

RREF

$$S=\left\{\begin{pmatrix}-2s-3t\\s\\t\\t\end{pmatrix}\middle|\, s,t\in\mathbb{R}\right\}$$

If we choose $s := -1$ and $t := 0$:

$$2\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}-1\cdot\begin{pmatrix}2\\10\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow 2\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}=\underline{\begin{pmatrix}2\\10\\4\end{pmatrix}}$$

If we choose $s := 0$ and $t := -1$:

$$3\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}-1\cdot\begin{pmatrix}-1\\-4\\-2\end{pmatrix}-1\cdot\begin{pmatrix}4\\19\\8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow 3\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}-1\cdot\begin{pmatrix}-1\\-4\\-2\end{pmatrix}=\underline{\begin{pmatrix}4\\19\\8\end{pmatrix}}$$

So, we can remove the 2nd and 1st vector without changing the span. The remaining vectors are linearly independent because:

$$a\cdot\begin{pmatrix}1\\5\\2\end{pmatrix}+b\cdot\begin{pmatrix}-1\\-4\\-2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{cc|c}1 & -1 & 0\\5 & -4 & 0\\2 & -2 & 0\end{array}\right]\sim\left[\begin{array}{cc|c}1 & -1 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right]\sim\left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right]$$

1st and 3rd column of above RREF

16.5 How we define the null space

$$A=\begin{bmatrix}1 & 2 & -1 & 4\\5 & 10 & -4 & 19\\2 & 4 & -2 & 8\end{bmatrix}.$$

$\mathrm{Null(A)}=\{\mathrm{x}^4\in\mathbb{R}^4|Ax=0\}$

Using this definition ^, we can easily check if a vector is in Null(A) or not:

$(1,0,0,0) \notin \text{Null(A)}$

because:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2,-1,0,0) \in \text{Null(A)}$

because

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This method is much easier than considering a spanning set and asking if a vector is a linear combination of two vectors in the spanning set.
We can fortunately describe spans as null spaces:

$$span\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$= \left\{ s \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; s \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ \text{z} \end{pmatrix} \text{ for some } s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = \begin{pmatrix} x \\ y \\ \text{z} \end{pmatrix} \text{ for some } s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; \left[\begin{array}{cc|c} 1 & 1 & x \\ 2 & 0 & y \\ 3 & 1 & z \end{array}\right] \text{ is consistent} \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y - 2x \\ 0 & 0 & z - y - x \end{array}\right] \text{ is consistent} \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; -x - y + z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} \text{x} \\ \text{y} \\ \text{z} \end{pmatrix} \middle| \; [-1 \quad -1 \quad 1] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0] \right\}$$

$$= \text{Null}[-1 \quad -1 \quad 1]$$

16.6 Extending linearly independent sets to a basis of $\mathbb{R}^n$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

So, a basis for the row space is $\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$.

If we want to extend this basis of $Row(A)$ to a basis of $\mathbb{R}^4$, we have to add two more vectors. The easiest way is to add vectors $(0 \ldots 0 \;\; 1 \;\; 0 \ldots 0)$ where the 1 is in a column in which the RREF didn't have a leading 1. In our case:

$$\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Then,

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&2&0&3\\0&0&1&-1\\0&1&0&0\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&2&0&3\\0&1&0&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\sim\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\text{RREF}}$$

There is a leading one in every column, so the vectors are linearly independent, and hence form a basis of $\mathbb{R}^4$.

$\dim(Row(A)) = rank(A)$

$\dim(Col(A)) = rank(A)$

$\dim(Null(A))$
$= \#\ of\ columns\ of\ A - rank(A)$

17 Bases and invertible matrices

17.1 Finding bases in general vector spaces

Problem:

Find a basis of the subspace W of $\mathbb{P}_3$:
$W = span\{3 + x + 4x^2 + 2x^3, 2 + 4x + 6x^2 + 8x^3, 1 + 3x + 4x^2 + 6x^3, -1 + 2x + x^2 + 4x^3\}$

Choose an ordered basis of $\mathbb{P}_3$, say $B = \{1, x, x^2 x^3\}$, and work with the coordinate vectors, which live in $\mathbb{R}^4$.

$$span\left\{\begin{bmatrix}3\\1\\4\\2\end{bmatrix}_B, \begin{bmatrix}2\\4\\6\\8\end{bmatrix}_B, \begin{bmatrix}1\\3\\4\\6\end{bmatrix}_B, \begin{bmatrix}-1\\2\\1\\4\end{bmatrix}_B\right\}$$

Run the row space algorithm:

$$\begin{bmatrix}3&1&4&2\\2&4&6&8\\1&3&4&6\\-1&2&1&4\end{bmatrix}\sim\begin{bmatrix}1&3&4&6\\0&-2&-2&-4\\0&-8&-8&-16\\0&5&5&10\end{bmatrix}\sim\begin{bmatrix}1&3&4&6\\0&1&1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\sim\underbrace{\begin{bmatrix}1&0&1&0\\0&1&1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{\text{RREF}}$$

Read off a basis and translate back to $\mathbb{P}_3$:

$$\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B\right\} \rightarrow \{1 + x^2, x + x^2 + 2x^3\}$$

17.2 Extending LI sets to bases in general vector spaces

Problem

Extend $\{1 + x^2, x + x^2 + 2x^3\}$ to a basis of $\mathbb{P}_3$. We use the same strategy as before. Consider $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B\right\}$ and add two more vectors:

$$\begin{array}{c} \text{old}\left\{\begin{array}{c}\\ \\ \end{array}\right. \\ \text{new}\left\{\begin{array}{c}\\ \\ \end{array}\right. \end{array}\begin{bmatrix}1&0&1&0\\0&1&1&2\\0&1&0&0\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\ \text{RREF}$$

The basis of $\mathbb{R}^4$ is:

$$\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}_B, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}_B\right\} \rightarrow \{1 + x^2, x + x^2 + 2x^3, x, x^3\}$$